Cassini - Huygens and Gravity Assist

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Introduction

Cassini – Huygens is a joint NASA, ESA and ASI project to, "explore the Saturnian system and all its elements: the planet and its atmosphere, rings and magnetosphere, and a large number of its moons, particularly Titan and the icy satellites" (European Space Agency). The Cassini Orbiter is the main spacecraft. The Huygens Probe, carried by Cassini to Saturn, was designed to detach and land on Saturn's moon Titan, to investigate the atmosphere and surface.

The Cassini-Huygens spacecraft was launched aboard a U.S. Titan IV-B launch vehicle. This is the largest and most powerful expendable launch vehicle used by NASA. Even using the Titan IV-B, it was not possible to for Cassini - Huygens to be sent on a conventional interplanetary trajectory to Saturn. NASA was forced to fly a circuitous route that included flybys of Venus (twice), Earth and Jupiter. This seems like an extremely complicated path; however, the route included what are called gravity-assist maneuvers at each flyby that were used to generate the large change in velocity that was required. In this paper I will explain why a conventional trajectory from Earth to Saturn was excluded, and how the gravity assist maneuvers made the mission possible.

Background

There is actually surprisingly little physics required to understand the orbital mechanics of the Cassini – Huygens mission to an approximation that shows why the conventional trajectory was excluded and why gravity assist was required. The following is a quick review of the most important concepts.

Orbits

According to Kepler, an orbit about the Sun is an ellipse with the Sun at one focus. The orbits of the planets we will be considering have very small eccentricities and so can be approximated as being circular. The Sun acts as a central force in the solar system; and so, according to Newton, $F = ma$ and,

$$\frac{Gmm_{\odot}}{r^2} = m\frac{v^2}{r}. \tag{1}$$

Here G is Newton's gravitational constant, m is the mass of the orbiting body, $m_{\odot}$ is the mass of the Sun, r is the radius of the orbit and v is the orbital velocity. Equation (1) implies that there is a specific velocity at which each planet must be moving in its orbit given by,

$$v = \sqrt{\frac{Gm_{\odot}}{r}}. \tag{2}$$

Energy

The energy of a body in a heliocentric orbit is given by the sum of its kinetic energy and potential energy. Recall that kinetic energy is given by,

$$K = \frac{1}{2} m v^2 . \tag{3}$$

In this expression, m is the mass of the body and v is its velocity. If the Sun is approximated as a point mass, the potential energy of a body in a solar orbit is given by,

$$U = -\frac{G m m_{\odot}}{r} . \tag{4}$$

Here G is the gravitational constant, m is the mass of the orbiting body, $m_{\odot}$ is the mass of the Sun, and r is the radius of the orbit. The total energy is defined as the sum of the kinetic and potential energies, therefore,

$$E = \frac{1}{2} m v^2 - \frac{G m m_{\odot}}{r} . \tag{5}$$

This expression can be rewritten in a form more suitable to orbital mechanics by factoring out the mass of the orbiting body, m.

$$E = m\left(\frac{1}{2} v^2 - \frac{G m_{\odot}}{r}\right) = mC \tag{6}$$

C (sometimes written E) is therefore the total energy of the orbiting body per unit mass.

Changing Orbits

Moving between two orbits implies transferring from an initial orbit involving an initial energy at a given radius, to a final orbit and energy at a new radius. From equation (6) it can be seen that the initial energy and final energies per unit mass are,

$$C_i = \frac{1}{2} v_i^2 - \frac{G m_{\odot}}{r_i} \tag{7}$$

$$C_f = \frac{1}{2} v_f^2 - \frac{G m_{\odot}}{r_f} . \tag{8}$$

In equation (7), C_i is the energy per unit mass of the initial orbit, v_i is the velocity of a body in the initial orbit, and r_i is the radius of the initial orbit. In equation (8), C_f is the energy per unit mass of the final orbit, v_f is the velocity of a body in the final orbit, and r_f is the radius of the final orbit. In both equations, G is Newton's gravitational constant and $m_{\odot}$ is the mass of the Sun. A change from an initial orbit to a final orbit therefore means that the total energy of the orbiting body must change by,

$$\Delta C = C_f - C_i . \tag{9}$$

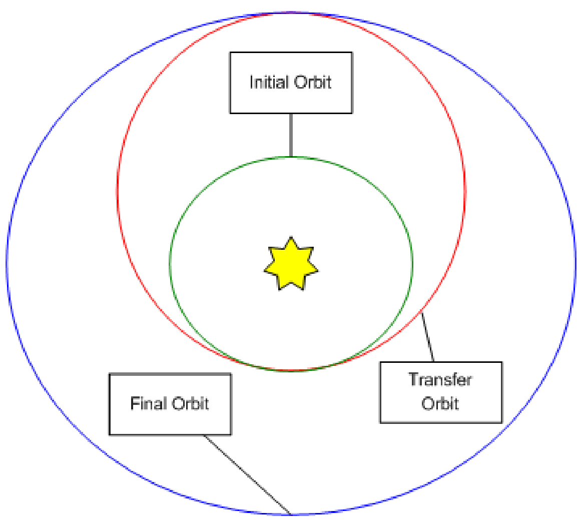
By combining equations (2), (7), (8) and (9), along with some algebra, it can be shown that,

$$\Delta C = \frac{Gm_{\odot}}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right). \tag{10}$$

Equation (10) represents the energy per unit mass required to change from an initial orbit of radius r_i to a final orbit of radius r_f .

Transfer Orbits

One of the most energy efficient ways to transfer from one orbit to another is via an elliptical transfer orbit. It is also one of the easiest ways to understand, conceptually. In the diagram below, an inner, initial orbit is depicted in green. A final orbit is depicted in blue. A transfer orbit between the two is simply a cotangential ellipse, with the Sun at one focus, that connects the initial and final orbits. One possible transfer orbit is shown in red in the diagram below.



Equation (10) defines the energy required to change from one orbit to another. A transfer orbit requires two such changes: first a body must move from the initial orbit to the transfer orbit; and then it must move from the transfer orbit into the final orbit. Just as in circular orbits, an elliptical transfer orbit has an associated energy,

$$C_T = \frac{1}{2}v^2 - \frac{Gm_{\odot}}{r} . \tag{11}$$

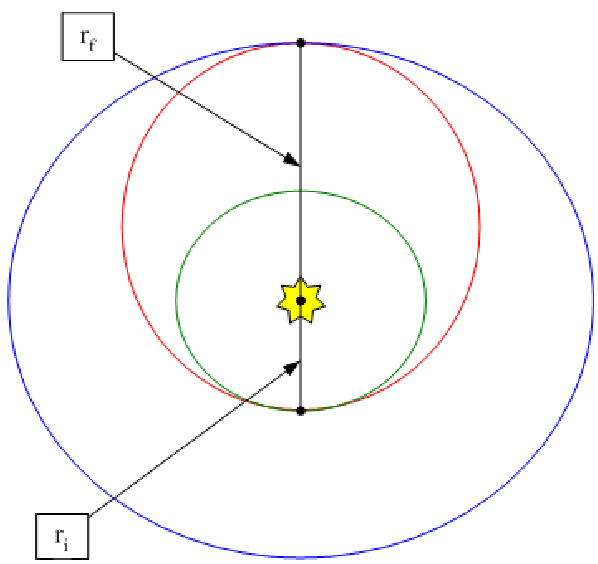
Since the orbit is not circular, r is taken as the distance of the orbiting body from the focus occupied by the Sun. For elliptical motion, the velocity in the orbit is given by,

$$v^2 = Gm_{\odot}\left(\frac{2}{r} - \frac{1}{a}\right). \tag{12}$$

Here v is the velocity of the body, G is Newton's gravitational constant, $m_{\odot}$ is the mass of the Sun, r is the distance to the occupied focus and a is the semimajor axis of the ellipse. Substituting equation (12) into equation (11) results in,

$$C_T = -\frac{Gm_{\odot}}{2a}, \tag{13}$$

where C_T is the energy associated with the transfer orbit. Interestingly, this shows that the energy of an elliptical orbit is purely a function of its semimajor axis.[1] In the diagram below, observe that the major axis of the ellipse is the sum of the radii of the inner orbit (r_i) and the outer orbit (r_f).



This means that the semimajor axis of the transfer ellipse can be written as,

$$a = \frac{1}{2}\left(r_i + r_f\right). \tag{14}$$

Substituting equation (14) into equation (13) yields,

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[1] For a circular orbit approximation, the semimajor axis may be replaced by the radius of the circular orbit and equations (7) and (8) may be recovered.

$$C_T = -\frac{Gm_{\square}}{r_i + r_f}\,. \tag{15}$$

Now, observe that equation (7), the energy of the initial orbit, can be written in a similar form,

$$C_i = -\frac{1}{2}\frac{Gm_{\square}}{r_i}\,. \tag{16}$$

Combining equations (15) and (16) to find the difference in energy between the orbits results in,

$$\Delta C_A = C_T - C_1 = -\frac{Gm_{\square}}{r_i + r_f} + \frac{1}{2}\frac{Gm_{\square}}{r_i}\,. \tag{17}$$

This is then the change in energy required for entry into the elliptical transfer orbit from the initial orbit. With a little algebra, equation (17) can be written,

$$\Delta C_A = -\frac{Gm_{\square}}{2r_i}\left(\frac{r_i - r_f}{r_i + r_f}\right). \tag{18}$$

Similarly, the energy change required to exit the transfer orbit and enter the final orbit is,

$$\Delta C_B = C_f - C_T = -\frac{Gm_{\square}}{2r_f}\left(\frac{r_i - r_f}{r_i + r_f}\right). \tag{19}$$

These changes in energy, ΔC_A and ΔC_B are due to changes in kinetic energy (in the case of a spacecraft, it will fire its rocket motor to achieve the change). We can therefore write,

$$\Delta C_A = \frac{1}{2}\left(v_1 - \Delta v_1\right)^2 - \frac{1}{2}v_1^2\,, \tag{20}$$

$$\Delta C_B = \frac{1}{2}v_2^2 - \frac{1}{2}\left(v_2 - \Delta v_2\right)^2 \tag{21}$$

Equating (20) and (18) gives,

$$\frac{1}{2}\left(v_1 - \Delta v_1\right)^2 - \frac{1}{2}v_1^2 = -\frac{Gm_{\square}}{2r_i}\left(\frac{r_i - r_f}{r_i + r_f}\right) \tag{22}$$

With some algebra, it can be shown that,

$$\Delta v_1 = \sqrt{\frac{Gm_{\oplus}}{r_i}} \left(\sqrt{\frac{2r_f/r_i}{1+r_f/r_i}} - 1 \right). \tag{23}$$

Here, Δv_1 is the change in velocity that a spacecraft must achieve in order to exit the initial orbit and enter the transfer orbit. There is a similarly derived expression for the change in velocity required to enter the final orbit from the transfer orbit,

$$\Delta v_2 = \sqrt{\frac{Gm_{\oplus}}{r_f}} \left(1 - \sqrt{\frac{2}{1+r_f/r_i}} \right). \tag{24}$$

The Tsiolkovsky Rocket Equation

The tools required to calculate the change in velocity required to move from one orbit to another have now been developed. The next step is to understand how to calculate the mass of fuel required in order to achieve a given change in velocity. The simplest way to do so is by using the Tsiolkovsky Rocket Equation,

$$\Delta v = v_e \ln\left(\frac{m_i}{m_f} \right). \tag{25}$$

Here, Δv is the total change in velocity that results from a mass of fuel being burned and expelled. The variable v_e is the velocity of the rocket exhaust in the rocket's reference frame, m_i is the initial total mass (rocket, payload and fuel combined), and m_f is the final mass (rocket and payload only, with all of the fuel burned and expelled).

The Tsiolkovsky Rocket Equation is derived from momentum conservation, the differential form of which is $mdv = v_e dm_{fuel}$. At the most abstract level, this says that there must be an increase in the momentum of the spacecraft equal to the momentum of the exhausted fuel. From the rocket equation it is easy to see that there are several ways to achieve large values of Δv:

- You can have a huge initial mass – an exponentially increasing mass of fuel as the required Δv rises;
- You can have a very small final mass (read payload);
- You can have a very large exhaust velocity.

The maximum realistic exhaust velocity of chemical rockets is about 4,500 meters per second. The highest exhaust velocity for a chemical propellant ever test-fired in a rocket engine was from a lithium, fluorine, and hydrogen rocket at 5,320 m/s. The space shuttle main engines have a v_e of 4,500 m/s, which is typical.

Analyzing a Cassini – Huygens Transfer Orbit

It can now be shown why the Cassini – Huygens mission was not possible using an elliptical transfer orbit connecting the orbit of the Earth to the orbit of Saturn. In such a transfer orbit, the spacecraft would need to change velocities to enter the transfer orbit, and then again to enter Saturn's orbit.[2] An estimate of the fuel requirements can be made using the required Δv and the Tsiolkovsky Rocket Equation.

Newton's gravitational constant, $G = 6.67\times10^{-11}\ N\cdot m^2/\text{kg}^2$, and the mass of the Sun, $m_{\odot} = 1.99\times10^{30}\,kg$. The radius of Earth orbit is 150,000,000 km and the radius of Saturn's orbit is 1,430,000,000 km. Putting these numbers into equations (23) and (24) results in,

$$\Delta v_1 \approx 12\times10^3\ m/s$$
$$\Delta v_2 \approx 3\times10^3\ m/s\,.$$

The sum of Δv_1 and Δv_2 is the change in velocity required to fly a spacecraft from Earth orbit to Saturn orbit, and is shown to be

$$\Delta v \approx 15\times10^3\ m/s\,.$$

The Tsiolkovsky Rocket Equation is used to determine fuel requirements. The Cassini orbiter weighs 2,150 kg and the Huygens probe weighs 350 kg (AeroSpaceGuide). First, solve the rocket equation for m_i ,

$$m_i = m_f e^{\frac{\Delta V}{v_e}} \tag{26}$$

Recall that Δv is the total velocity change requirement of the transfer orbit, and v_e is the rocket exhaust velocity. The value m_i is the total weight of the vehicle with fuel, and m_f is the empty weight. If we use the typical value of 4,500 for v_e , and use the combined Cassini – Huygens weight of 2,500 kg as the empty weight (neglecting the mass of tankage), we find,

$$2500e^{\frac{15000}{4500}} = 70079kg\,. \tag{27}$$

In order to fly an Earth to Saturn mission using an elliptical transfer orbit, the total weight of the Cassini – Huygens and fuel would need to be about 70,000 kg. That is significantly heavier than an American M1 Abrams main battle tank. This 70,000 kg

[2] More precisely, the spacecraft would have to enter an orbit *of* Saturn, not enter Saturn's orbit. This is a more complicated analysis but the result is ultimately the same.

spacecraft must be launched into Earth orbit before it can even begin the transfer orbit, though.

NASA has six types of launch vehicles available. Their payload capacities are given in the following table (National Aeronautics and Space Administration).

Name	**Payload**
Pegasus	454 kg
Taurus	1350 kg
Athena II	1896 kg
Atlas V	8670 kg
Delta	12757 kg
Titan IVB/Centaur	21727 kg

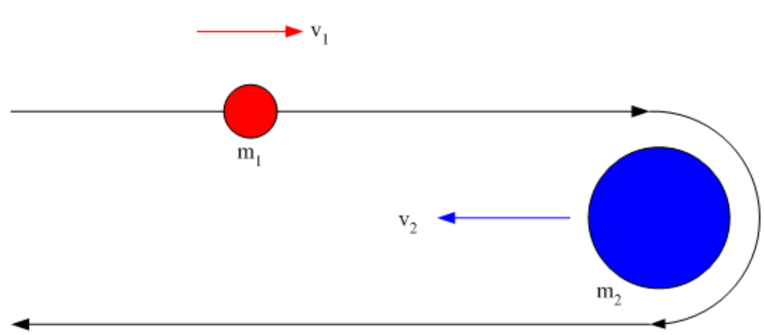
There is no launch vehicle in the NASA stable that is remotely close to being able to lift a spacecraft weighing 70,000 kg, and so the conventional transfer orbit solution must be excluded.

The Gravity Assist Approach

While it is not possible to lift a Cassini – Huygens mission that would use a conventional trajectory, there is an alternative. Several robotic spacecraft have taken advantage of the gravity assist technique to reach distant destinations. Using gravity assist, a spacecraft can interact with a planet to add or subtract momentum, and therefore increase or decrease the energy and speed of its orbit. As shown in equation (10), a change in energy corresponds to a change in orbit radius. The gravity assist method is straightforward and, in essence, the spacecraft "steals" momentum from a planet.

An Extreme Example of Gravity Assist

Imagine an extremely eccentric hyperbolic orbit. In fact, picture an orbit so eccentric it looks like a course reversal. This situation is illustrated below. Consider a spacecraft, labeled m_1 in the illustration, traveling at a velocity v_1 which encounters a planet m_2 moving in the opposite direction at velocity v_2.



In this interaction, conservation of momentum must apply, so that

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Solving for the final velocity of the planet, v_{2f}, one can see,

$$v_{2f} = \frac{m_1}{m_2} v_{1i} - \frac{m_1}{m_2} v_{1f} + v_{2i}.$$

Since the mass of the spacecraft m_1, is small compared to the planet m_2, $\frac{m_1}{m_2} \approx 0$ and

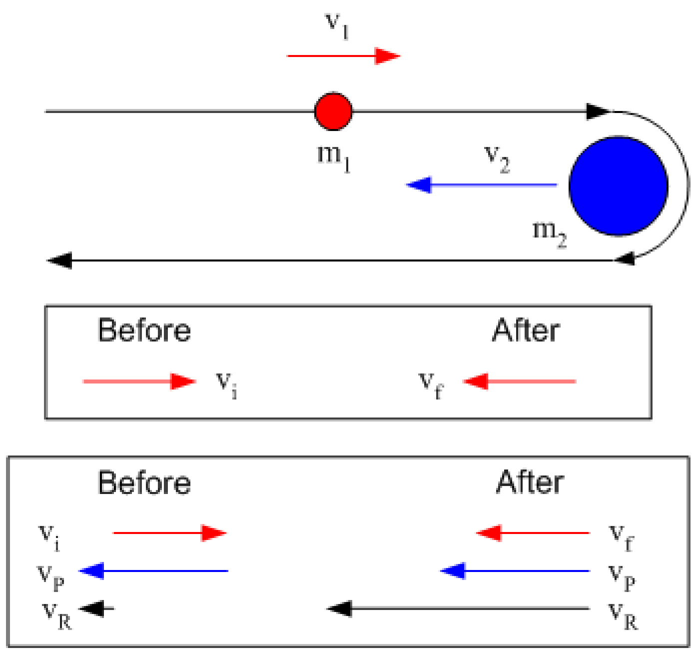
$$v_{2f} \approx v_{2i}.$$

Moving into the rest frame of the planet, the spacecraft appears to approach the planet at velocity $v_1 + v_2$. In this frame, it will depart the planet at an equal velocity $v_1 + v_2$ in the opposite direction. In the frame of reference of the Sun, however, the planet is moving at velocity v_2. In order to convert from the planet rest fame back to the Sun frame, the velocity of the planet must be added to the velocity of the spacecraft. This means that from the point of view of a "stationary" solar system, the spacecraft started the planetary encounter with a velocity of $v_1 + v_2$ and finished with a velocity of $(v_1 + v_2) + v_2 = v_1 + 2v_2$. In the frame of reference of the solar system, the speed of the spacecraft has changed from $v_1 + v_2$ to $v_1 + 2v_2$.

It is enlightening to examine this result using vectors. The following illustration is of the same interaction, but shows the velocity vectors as seen in the planet rest frame and in the Sun frame. From the point of view of the planet's rest frame, as the spacecraft moves in from the left, it has a velocity vector v_i. This is shown in the upper box in the illustration as the "before" vector. During the encounter, v_i is rotated by 180° as shown in the upper box "after" vector, v_f. In the rest frame of the planet, the magnitude of this vector is unchanged, only the direction changes. The spacecraft exits the encounter in the opposite direction.

In order to convert the encounter to the Sun frame, the velocity of the planet must be added. In the lower box of the illustration, the velocity vectors of the spacecraft and the planet (v_p) are added "head to tail" giving the resultant vector v_R. The resultant vector is interpreted as the velocity of the spacecraft in the Sun reference frame.

Looking at the lower, resultant vectors, it is easy to see that the illustrated interaction can actually represent a slowly moving spacecraft (v_R before) being overtaken by a fast moving planet and speeding up as a result of the interaction (v_R after).



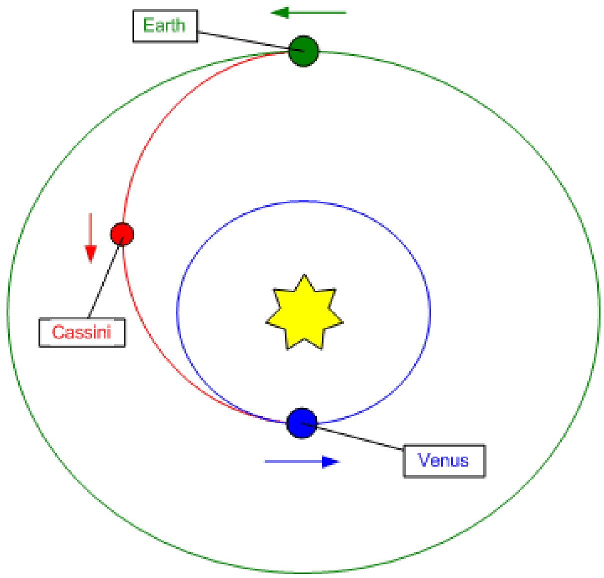
During this kind of encounter, a spacecraft will receive a boost in speed known as a gravity assist. The increase in speed may be understood by looking at the trajectory change of the spacecraft as a rotation of its velocity vector into an orientation closer to parallel with the planet's velocity vector. In the case of the illustration above, the spacecraft velocity vector started out antiparallel to the planet's velocity vector and ended up being parallel. This represents the greatest amount of speed increase possible in an interaction – a rotation through 180º. In real missions the encounter is not a course reversal, however, the situation is more subtle.

Realistic Form of Gravity Assist

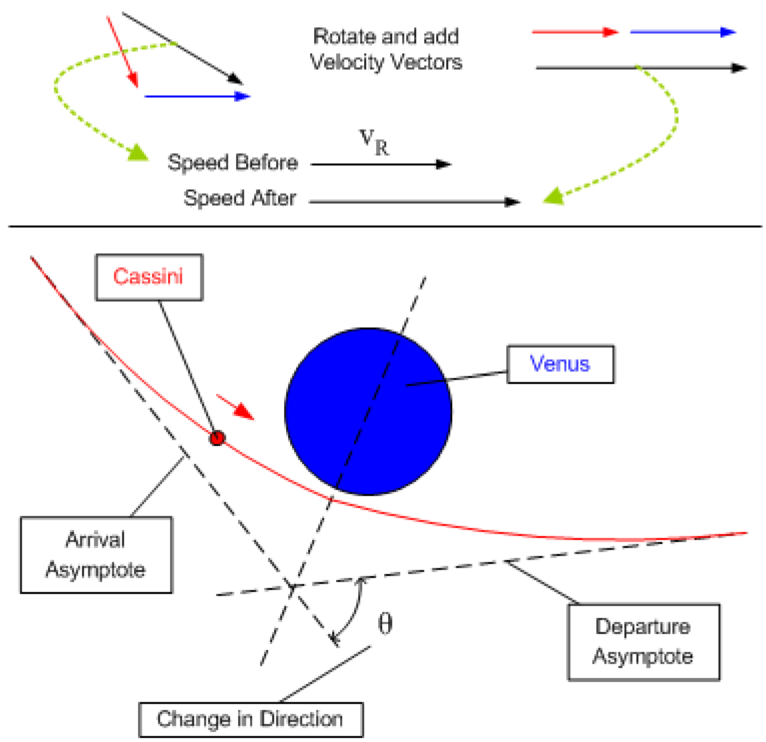
In a typical mission requiring a gravity assist maneuver, the goal is to send a spacecraft to an outer planet that, for some reason – usually cost, is not reachable via a conventional elliptical transfer orbit. Ironically, gravity assist missions requiring large increases in velocity typically begin by slowing the spacecraft down to enter an elliptical transfer orbit to an inner planet. Inner planets are chosen since they have greater orbital velocity and can therefore transfer more momentum during the encounter. Recall that

$$v = \sqrt{\frac{GM_{\odot}}{r}},$$

so the smaller the orbital radius, the greater the orbital velocity. The illustration below schematically shows a transfer orbit from Earth (outer, green orbit) to Venus (inner, blue orbit).



If we were to zoom in on the Venus encounter depicted above, we would see something like the following illustration. In the rest frame of Venus, it appears that the spacecraft is approaching on a hyperbolic trajectory that is essentially parallel to the left (arrival) asymptote and will exit along the right (departure) asymptote. In the Venus rest frame, it appears as if the spacecraft velocity vector has been rotated by the angle θ with its speed remaining unchanged.

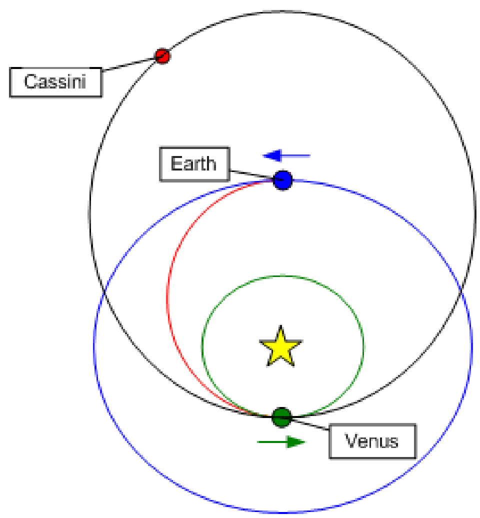


Just as in the extreme example above, to find velocities relative to the solar system, Venus' velocity vector must be added to both the initial and final spacecraft velocity vectors. This is shown in the top of the illustration. As the spacecraft velocity vector is rotated through the angle θ, it becomes closer to parallel with the Venus velocity vector. The length of the resultant vector therefore must increase, and the speed of the spacecraft in the reference frame of the solar system must increase – a gravity assist has occurred.

The Cassini – Huygens Gravity Assist Maneuvers

In the Cassini – Huygens mission, the first gravitational assist was given by Venus, just as in the example above. In this maneuver, the Cassini – Huygens probe gained roughly 6 km/s relative to the sun.

The mission planners used this gravity assist to send the spacecraft into an elliptical orbit that actually went outside the radius of Earth orbit, and returned for a second gravity assist maneuver at Venus.[3] This elliptical orbit is shown in the illustration below.

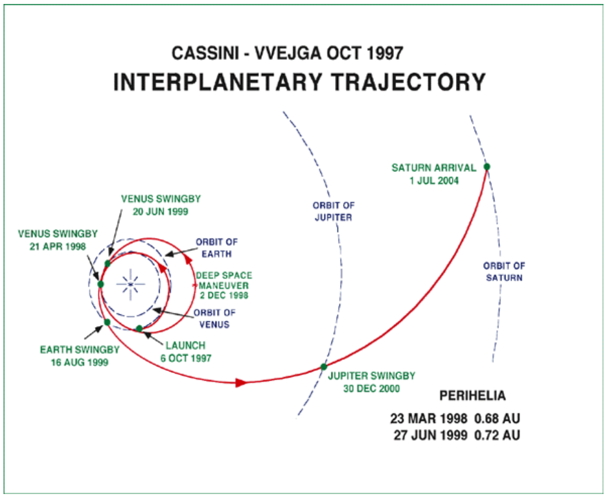


In the second gravity assist maneuver, the Cassini – Huygens probe gained another roughly 7 km/s. This was followed by a third gravity assist during an Earth flyby in which the spacecraft gained yet another 6 km/s. A fourth gravity assist occurred at a rendezvous with Jupiter which contributed a further 2 km/s.

These four gravity assist maneuvers correspond to a total Δv of 21,000 m/s. Note that this is more than the estimate of 15,000 m/s required for the Earth to Saturn mission since the gravity assists need to transfer the spacecraft from *Venus* to Saturn and not Earth to Saturn.

The entire route taken by Cassini - Huygens is shown in the following illustration.

[3] The spacecraft did not, in reality return to exactly the same spot in Venus' orbit. It is shown this way here for clarity.



The Venus, Venus, Earth, Jupiter Gravity Assist trajectory, known as VVEJGA to NASA mission planners, allowed Cassini - Huygens to generate enough velocity change to accomplish its mission. Since gravity assist requires no expenditure of fuel, the Cassini – Huygens spacecraft only had to take enough fuel to perform the final Saturn orbit entry maneuver. This was a significant maneuver in itself, and corresponded to a main rocket burn of 95 minutes that consumed most of the roughly 3,000 kg of propellant that the spacecraft carried.

Conclusion

Without using the gravity assist technique, the Cassini – Huygens spacecraft would have been required to carry almost 70,000 kg of fuel to get from Earth to Saturn. This is almost four times the maximum payload of the largest rocket in the NASA inventory.

By using gravity assist to achieve most of the velocity changes required for the mission, the designers were able to reduce the fuel requirements to 3,000 kg. Even though Cassini – Huygens is still one of the largest, heaviest and most complex interplanetary spacecraft ever produced, use of gravity assist maneuvers put its initial mass well within the reach of the Titan IVB/Centaur rocket that was ultimately used.

References

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